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DESIGN OF FIRST ORDER DIFFERENTIATOR WITH PARALLEL ALL-PASS STRUCTURE

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Abstract. *In this paper a new method for design of the first order differentiator is presented. The proposed differentiator consists of two parallel branches, i.e. direct path and IIR all-pass filter. The described design method allows one to obtain solution with minimum mean relative error at the desired region by controlling the ratio of phase response extremes. A small relative magnitude error, as well as a low phase error, at low frequencies is condition for good time domain behaviour. The obtained differentiator can be realized by means of only two multipliers, hence being a good choice for real time applications. The proposed solution provides a lower magnitude error than several known differentiators with similar phase error.*

Key words: *Recursive differentiator, all-pass filters, approximately linear phase, parallel structure, minimum multiplications*

1. INTRODUCTION

Digital differentiators are an important class of digital filters which can be used in almost all engineering disciplines including control systems, communications, radar and signal processing applications. Also, digital differentiators are widely applied in the field of biomedical signal processing, speech and image processing and seismic systems. With this broad range of applications, the design of digital differentiators is extensively researched over a few decades. Therefore, the design of differentiators is of significant interest.

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The frequency response of an ideal digital differentiator is equal to $j\omega$, where $j = \sqrt{-1}$ and ω is the angular frequency in radians. In general, a differentiator should have a linear phase response over the whole [0, π] frequency band to avoid the phase distortion. Differentiators having a perfectly linear phase response can be easily designed using FIR filters. Although causal and stable IIR differentiators do not have exact linear phase responses, they usually have lower filter orders and lower group delays compared to FIR counterpart differentiators. In applications where a linear phase is not required, IIR differentiators are more attractive than FIR differentiators for two main reasons. Firstly, they can satisfy given filter specifications with a much lower filter order, thereby reducing the computational requirement or the hardware complexity if hardware implementation is considered and, secondly, they usually have a much lower group delay thereby resulting in a lower system delay.

Different approaches to design a digital differentiator have been proposed in the literature. The conventional approach to the IIR differentiator design is obtained by inverting the transfer function of the IIR integrator. The other design algorithms are based on approximation, optimization and interpolation techniques. Paper [1] has reported on the design where the passband phase response linearity error is minimized. Results of that research showed that the maximum passband amplitude response relative error and phase response error, for the same passband amplitude response error constraints, is lower compared to the differentiators designed using the state-of-the-art competing methods. In 2019, Goswami *et. all.* proposed a method which interpolates the bilinear transform and rectangular transform fractionally [2]. The genetic algorithm is used for optimization of unknown variables. This approach also has a better magnitude response than all state-of-the-art designs. Recursive digital differentiators have been designed by Al-Alaoui [3, 4], who proposed a novel approach to design a first-order digital differentiator by interpolating standard trapezoidal rule and rectangular rule linearly. Results showed linearity up to 0.7 of the full normalized Nyquist frequency range. Furthermore, fractional bilinear transformation, proposed by Pei and Hsu, is another method for designing a first-order digital differentiator [5].

An important aspect of the paper [6] is that the exploration is focused on the design of first-order differentiators and integrators wider operating frequency bandwidth and linearity using bilinear transformations. Milić *et. all.* developed the structure of the overall differentiator composed of an approximately linear-phase all-pass filter, FIR poly-phase sub-filters, and a pure delay element [7]. Novelty of the proposed method in [8] lies in the fact that the wideband differentiator designed using lattice wave digital filter (LWDF) system incorporates all of the advantageous properties of LWDF with utilization of minimum hardware and also competent results are obtained using optimization techniques. Recently, Upadhyay introduced the designs which have similar magnitude responses with different phase characteristics. The lower percentage relative error for magnitude response over wideband, only with second-order systems makes these designs suitable for real-time applications [9, 10]. In [11], Ngo has also proposed a wideband digital integrator of order three which is based on the Newton-Cotes integration rule. It is noticed that these digital differentiators have nearly 7% or more relative error over the whole Nyquist band. Therefore, the design of wideband digital differentiators is the main issue in the current research environment. A new approach to the design of the nearly-linear phase infinite impulse response low-pass differentiators using a parallel all-pass structure is discussed in [12]. Comparison with the existing nearly-linear phase infinite impulse response low-pass differentiators shows that the low-pass differentiators designed using the proposed method usually require less multiplications. Also using the parallel all-pass structure, full-band differentiator whose magnitude response approximates the ideal one in the weighted Chebyshev sense and the phase response is a nearly-linear function of frequency at low frequencies, as given in paper [13]. A comprehensive analysis of the hardware complexity of different configurations for the realization of approximately linear phase filters is presented in [14]. The hardware complexity for the realization of the parallel all-pass structure is compared to the standard elliptic filters with the adequate group delay corrector in cascade. Both considered filters are designed to have the same cut-off frequency and magnitude approximation error, as well as the same maximum group delay error in all pass-bands. Conventional digital differentiators work efficiently up to low the frequency region only. The main purpose of the papers [15, 18] is to design and implement the first order Al-Alaoui differentiator at microwave frequencies.

This paper presents a method for designing an infinite impulse response digital differentiator with a nearly linear phase and magnitude responses. The proposed configuration outperforms existing differentiators of the same order in terms of the magnitude response. The novelty of the presented design is the introduction of the parallel all-pass structure resulting in the first order full-band differentiator transfer function requiring only two multiplications. The amplitude characteristic of proposed differentiators directly depends on all-pass network phase response and it is clear that the design can be achieved just through the all-pass filter phase approximation.

The approach for the design of recursive full-band digital differentiators using parallel all-pass structure is discussed in this paper. The design is realized using the parallel connection of direct path (zero order delay line) and the first order IIR all-pass filter. The magnitude response of a designed full-band differentiator approximates the ideal one in the weighted Chebyshev sense. The opposite of that, phase response linearity of the proposed IIR full-band differentiator cannot be controlled, while it is a nearly-linear function of frequency at low frequencies. The proposed design procedure is very efficient and the solution is obtained after only a few iterations.

2. Realization Structure

The frequency response of an ideal digital differentiator is described with:

$$H_{Ideal}(e^{j\omega}) = j\omega e^{-j\tau\omega} = j\omega e^{j\varphi_{Ideal}(\omega)}\ , \tag{1}$$

where τ represents the average group delay. It is obvious that magnitude characteristic is the linear function with zero value for $\omega = 0$ and reaching π at $\omega = \pi$. It holds:

$$\varphi_{Ideal}(0) = \frac{\pi}{2},\ \ \varphi_{Ideal}(\pi) = -(\tau - 0.5)\pi\ , \tag{2}$$

taking into account that the phase is linear with a slope -τ.

The proposed first order digital IIR differentiator could be realized as the parallel connection of direct path and the first order all-pass filter as shown on Fig. 1.

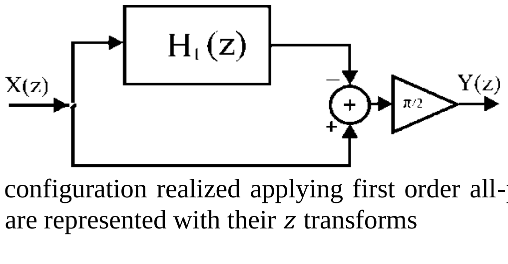

Fig. 1 Differentiator configuration realized applying first order all-pass filter. Input and output signals are represented with their z transforms

At the output of the composite filter, two branches feed the subtractor, as shown in Fig. 1. to provide the magnitude characteristic given with:

$$H_d(z) = \frac{Y(z)}{X(z)}, \quad \left|H_d(e^{j\omega})\right| = \pi\left|\sin\frac{\varphi_1(\omega)-\varphi_2(\omega)}{2}\right| = \pi\left|\sin\frac{\varphi_1(\omega)}{2}\right|, \tag{3}$$

where $\varphi_2(\omega)$ is the phase of the second branch (with zero value in our case) and $\varphi_1(\omega)$is the first order all-pass filter phase. The multiplier $\pi/2$ is required to obtain gain of π at $\omega = \pi$ hence magnitude response of the proposed differentiator matches that of an ideal one at DC and Nyquist frequencies. The phase of the proposed filter is:

$$\varphi_d(\omega) = \frac{\varphi_1(\omega)+\varphi_2(\omega)+\pi}{2} = \frac{\varphi_1(\omega)+\pi}{2}\ . \tag{4}$$

The phase of the first order IIR all-pass filter fulfils:

$$\varphi_1(0) = 0, \quad \varphi_1(\pi) = -\pi\ . \tag{5}$$

So, the phase of the differentiator would satisfy:

$$\varphi_d(0) = \frac{\pi}{2}, \quad \varphi_d(\pi) = 0\ , \tag{6}$$

and the average group delay would be:

$$\tau = \frac{\varphi_d(0)-\varphi_d(\pi)}{\pi} = 0.5\ , \tag{7}$$

given in samples. All previous information points to:

$$\varphi_d(0) = \varphi_{Ideal}(0), \quad \varphi_d(\pi) = \varphi_{Ideal}(\pi)\ , \tag{8}$$

i.e. zero approximation error at DC and the Nyquist frequency. Due to the straightforward dependence of the magnitude of the parallel all-pass structure and phases of filters from two branches, given with Eq. (3), to reach zero magnitude error, the stable all-pass filter needs to have the phase of shape

$$\varphi_{Ideal}(\omega) = -2\arcsin\frac{\omega}{\pi}\ , \tag{9}$$

which is not a linear function of frequency. We can conclude at this point that a compromise between magnitude and phase error will be basic for the approximation problem definition.

The transfer function of the all-pass filter is:

$$H_1(z) = \frac{a_1 + z^{-1}}{1+a_1 z^{-1}}\ , \tag{10}$$

satisfying:

$$H_1(e^{j\omega}) = e^{j\varphi_1(\omega)}, \quad \left|H_1(e^{j\omega})\right| = 1, \quad \angle H_1(e^{j\omega}) = \varphi_1(\omega)\ , \tag{11}$$

hence, the transfer function of the proposed differentiator is:

$$H_d(z) = \frac{\pi}{2}\left[1-\frac{a_1+z^{-1}}{1+a_1 z^{-1}}\right] = \frac{(1-a_1)\pi}{2}\frac{1-z^{-1}}{1+a_1 z^{-1}}\ . \tag{12}$$

The magnitude characteristic of the differentiator is given with:

$$\begin{aligned}\left|H_d(e^{j\omega})\right| &= \left|\frac{(1-a_1)\pi}{2}\right|\sqrt{\frac{(1-\cos\omega)^2+\sin^2\omega}{(1+a_1\cos\omega)^2+a_1^2\sin^2\omega}} = \\ &= \frac{(1-a_1)\pi}{2}\sqrt{\frac{2(1-\cos\omega)}{1+a_1^2+2a_1\cos\omega}},\end{aligned} \tag{13}$$

since the stable all-pass filter has all poles inside the unit circle ($|a_1|\leq 1$). Starting from the ideal all-pass filter's phase given with Eq. (9) and allowing the maximum phase deviation *Deg* given in degrees, it is easy to predict the region where characteristics of a real filter could be located:

$$\begin{aligned}\varphi_{1r} &= \varphi_{1Ideal}(\omega_i) + \frac{\pi Deg(-1+2rand(1,1001))}{180} \\ \omega_i &= \frac{(i-1)\pi}{1000},\quad i=1,2,\ldots,1001\end{aligned} \tag{14}$$

Results are depicted in Fig. 2, choosing the allowed error *Deg=7* degrees. Yellow regions correspond to the differentiator and the green one shows the all-pass filter's phase. The dashed black lines indicate ideal differentiator characteristics.

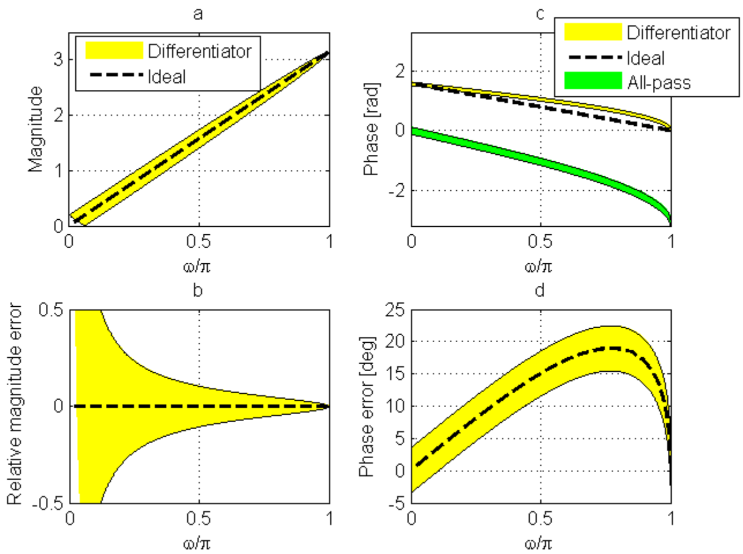

Fig. 2 (a) Possible areas for magnitude and (c) phase of first order differentiator with (b) corresponding relative magnitude error and (d) phase error for maximum all-pass filter phase error of 7 degrees

The same maximum all-pass filter phase error at all frequencies provides slightly higher magnitude error at low frequencies but induces a significant relative error in this area. It leads to the conclusion that a design needs to provide as low as possible phase error at low frequencies to ensure a permissible relative error at frequencies where input signal spectrum components are significant. As expected, the existence of non-linear term in the ideal phase will contribute to an enlarged differentiator phase error at higher frequencies according to the nature of arcsin (ω) function. In the ideal case (to remind, the term ideal corresponds to the case where an ideal magnitude is achieved) maximum phase error reaches 18.9 degrees. The designed filter could lower this value letting decreased all-pass phase error at the cost of the magnitude error increase.

3. Initial Solution and Iterative Approach in Filter Design

The phase of N^{th} order IIR all-pass filter is:

$$\varphi_N(\omega) = -N\omega + 2\arctan\frac{\sum_{i=1}^{N} a_i \sin(i\omega)}{1+\sum_{i=1}^{N} a_i \cos(i\omega)}\,. \tag{15}$$

which for N=1 reduces to:

$$\varphi_1(\omega) = -\omega + 2\arctan\frac{a_1 \sin\omega}{1+a_1\cos\omega}\,, \tag{16}$$

and the phase of differentiator becomes:

$$\varphi_d(\omega) = \frac{\pi}{2} - \frac{\omega}{2} + \arctan\frac{a_1 \sin\omega}{1+a_1\cos\omega}\,, \tag{17}$$

while the magnitude has the value:

$$\left|H_d(e^{j\omega})\right| = \pi\left|\sin\left(-\frac{\omega}{2} + \arctan\frac{a_1 \sin\omega}{1+a_1\cos\omega}\right)\right|. \tag{18}$$

In all published papers authors examine the relative magnitude error:

$$\delta_r(\omega) = \frac{\pi}{\omega}\left|\sin\left(-\frac{\omega}{2} + \arctan\frac{a_1 \sin\omega}{1+a_1\cos\omega}\right)\right| - 1\,, \tag{19}$$

rather than the absolute magnitude error. The differentiator design comes down to adequate determination of a coefficient a_1 solving the all-pass filter's phase approximation problem. In the nonlinear phase approximation, the all-pass phase error curve has one more extremum (two in our case) compared to the filter's order. Hence, one could form a system of equations to, except filter coefficient, one more parameter be optimized. Between two extrema, there exist one frequency, denoted as ω_z, where the relative magnitude error has zero value $\delta(\omega_z) = 0$:

$$\pi\left|\sin\left(-\frac{\omega_z}{2} + \arctan\frac{a_1 \sin\omega_z}{1+a_1\cos\omega_z}\right)\right| = \omega_z\,. \tag{20}$$

Let it be $\omega_z = \pi/2$, then Eq. (20) becomes:

$$\left|\sin\left(-\frac{\pi}{4} + \arctan a_1\right)\right| = 0.5\,, \tag{21}$$

to give $a_1^* = 0.268$ as the initial solution which provide a good starting point for the iterative procedure for the filter design. The relative magnitude error and the all-pass filter's phase error are displayed in Fig. 3.

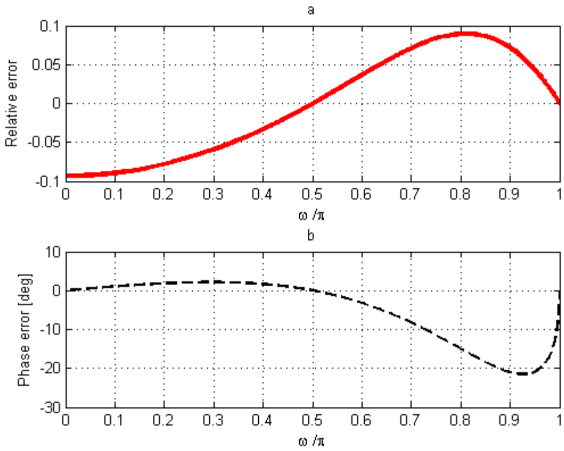

Fig. 3 (a) Relative magnitude error of differentiator and (b) phase error of all-pass filter for initial solution

Note that the obtained initial solution has an almost equiripple relative error. There are one maximum and one minimum at phase error curve with unequal values. As will be seen later, the successful design can be achieved by controlling the ratio of this two phase error extremes.

The equiripple magnitude error could be reached by solving the system of two equations:

$$\frac{\left|H_d(e^{j\omega_i})\right| - \omega_i}{\omega_i} = (-1)^i \delta_r, \quad i = 1, 2, \tag{22}$$

where δ_r is the maximum relative error and ω_i are frequencies of relative error extrema. After substituting the differentiator's magnitude function with the truncated Taylor series, system of Eq. (22) becomes:

$$\begin{bmatrix} \left.\dfrac{\partial\left|H_d(e^{j\omega_1})\right|}{\partial a_1}\right|_{a=a_1^*} & \omega_1 \\ \left.\dfrac{\partial\left|H_d(e^{j\omega_2})\right|}{\partial a_1}\right|_{a=a_1^*} & -\omega_2 \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta\delta_r \end{bmatrix} = \begin{bmatrix} \omega_1(1-\delta_r) - \left|H_d(e^{j\omega_1})\right| \\ \omega_2(1+\delta_r) - \left|H_d(e^{j\omega_2})\right| \end{bmatrix}, \tag{23}$$

with values of the differentiator magnitude and corresponding derivatives obtained using the starting value a_1^*.

$$\frac{\partial\left|H_d(e^{j\omega})\right|}{\partial a_1} = -\frac{\pi}{2}\cdot\sqrt{\frac{2(1-\cos\omega)}{1+a_1^{\,2}+2a_1\cos\omega}}\cdot\left(1+\frac{1-a_1}{2}\cdot\frac{a_1\cos\omega}{1+a_1^{\,2}+2a_1\cos\omega}\right). \tag{24}$$

The solution of system (23) is the column vector of increments and it allows one to iteratively obtain new values:

$$a_1 = a_1^{\,*} + \Delta a_1,\quad \delta_r = \delta_r^{\,*} + \Delta\delta_r\,, \tag{25}$$

which are initial solution[s] for the next iterative step. The procedure continues until the maximal absolute value of increments is larger than the predefined small number (in given examples value 10^{-10} has been chosen).

4. Evaluation of Differentiator Performance

To estimate the overall differentiator performance several parameters are calculated, namely the magnitude mean relative error (MRE_L, MRE_{1000}=MRE):

$$MRE_L = \frac{1}{L}\sum_{i=1}^{L}\left|\frac{\left|H_d(e^{j\omega_i})\right|}{\omega_i}-1\right|, \tag{26}$$

and the corresponding maximum relative error δ_r , phase mean error:

$$MPE_L = \frac{1}{L}\sum_{i=1}^{L}\left|\varphi_d\left(\omega_i\right)-\varphi_{Ideal}\left(\omega_i\right)\right|, \tag{27}$$

with the adherent maximum phase error δ_p. *MRE* and *MPE* in all examples have been calculated on a discrete set of equidistant frequency points $\omega_i = \pi_i / 1000$, $i = 1,2,...,1000$.

The equiripple relative magnitude error solution is presented in Fig. 4 with $a_1 = 0.2665$, $\delta_r = 0.0903$. The maximum phase error has value of 15.4569 degrees and the maximum all-pass phase deviation is 21.5626 degrees as given in Table 1. The other extremum of the all-pass phase error has the value 2.0368, located at low frequencies. As expected, a higher all-pass phase error contribute to the differentiator phase error lower than “*ideal*” 18.9 degrees.

This phase improvement is paid with an increased relative error near the Nyquist frequency where the all-pass phase extremum is located. The relative error near zero frequency demands a very small all-pass phase fluctuation as shown in Fig. 4(b).

Table 1 Parameters of differentiator designed by equiripple relative error

δ_r	*MRE*	MRE_{500}	$\delta_p\,[deg]$	*MPE*	MPE_{500}	a_1
0.0903	0.0581	0.0589	15.4569	9.7902	8.5018	0.266513

Instead of a design based on the relative error control, the solution is achievable through the all-pass phase error adjustment. The system described by Eq. (23) becomes:

$$\begin{bmatrix} \left.\dfrac{\partial\varphi_1(\omega_1)}{\partial a_1}\right|_{a=a_1^*} & 1 \\ \left.\dfrac{\partial\varphi_2(\omega_2)}{\partial a_1}\right|_{a=a_1^*} & -\alpha \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta\delta_p \end{bmatrix} = \begin{bmatrix} -\varepsilon - 2\arcsin\dfrac{\omega_1}{\pi} - \varphi_1(\omega_1) \\ \alpha\varepsilon - 2\arcsin\dfrac{\omega_2}{\pi} - \varphi_1(\omega_2) \end{bmatrix}, \tag{28}$$

where the parameter $\alpha > 1$ gives information how many times second extremum of the all-pass phase error is greater than the first one (ε). Such approach is intuitive, the second extremum at high frequency increases if α is increased, but the first extremum at low frequency decreases and guarantees a better magnitude of the differentiator exactly at frequencies where spectral components of the input signal are significant. Obtained results are given in Figs. 5-8.

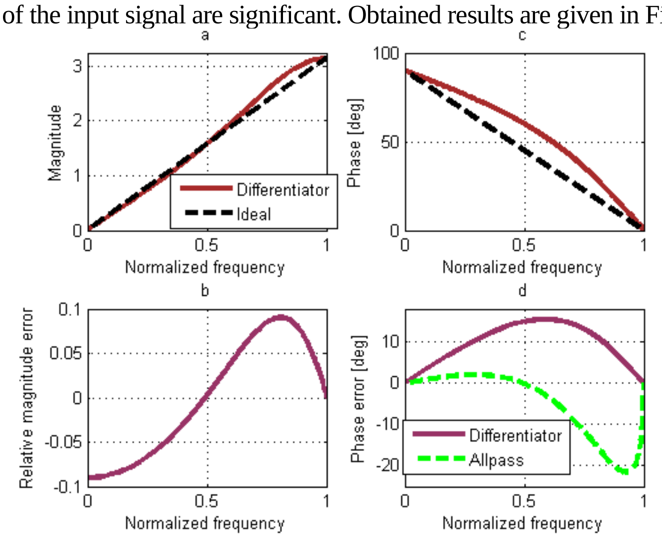

Fig. 4 (a) Magnitude, (b) relative error, (c) phase and (d) phase error of first order differentiator with equiripple relative error

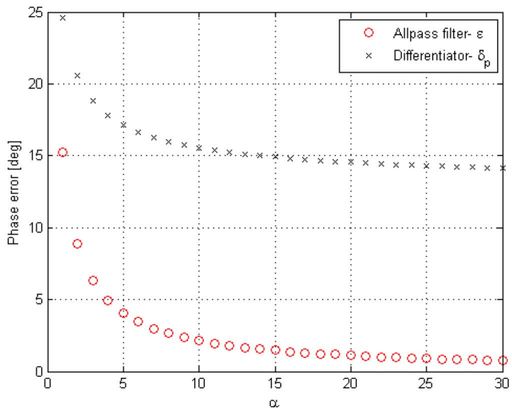

Fig. 5 Phase error of all-pass filter and differentiator for different values of parameter α

The higher ratio of two all-pass error extrema provides a lower differentiator phase error. It is evident that even for large α, this error is still higher than 14 degrees as shown in Fig. 5.

Increasing of parameter α lower relative error but this trend stops about α=11 as given in Fig. 6.

In Fig. 5, peak values of phase are given while Fig. 7 corresponds to average ones. No matter the second phase error extremum increase, its influence exists only at a narrow band at high frequencies, the mean phase error decrease. By comparing MRE_{1000} and MRE_{500} it can be concluded that all solutions exhibit a lower phase deviation at low frequencies.

From Fig. 8, one can conclude that *MRE* could be improved to some extent by increasing the second phase error extremum until saturation occurs. At low frequencies the mean relative error steadily improves by the increased α (see Table 2).

Table 2 Parameters of differentiator designed by minimization of MRE_{500}, α =34.9

δ_r	*MRE*	MRE_{500}	$\delta_p\,[deg]$	*MPE*	MPE_{500}	a_1
0.104117	0.05052	0.02539	14.0178	8.88524	7.82163	0.24222

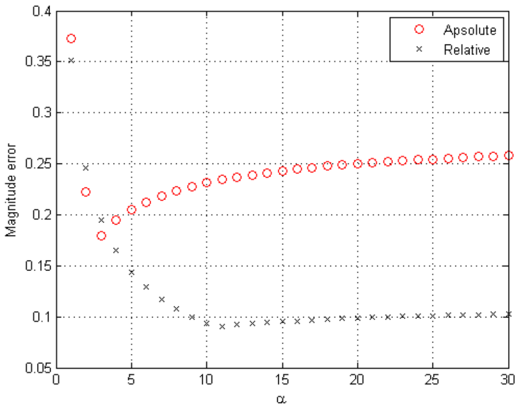

Fig. 6 Magnitude error of differentiator for different values of parameter α

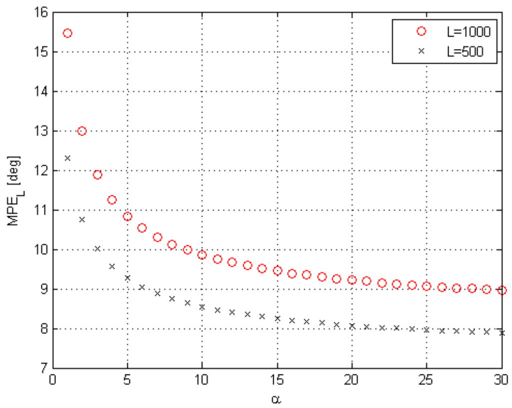

Fig. 7 Mean phase error for different values of parameter α for L=500 and L=1000

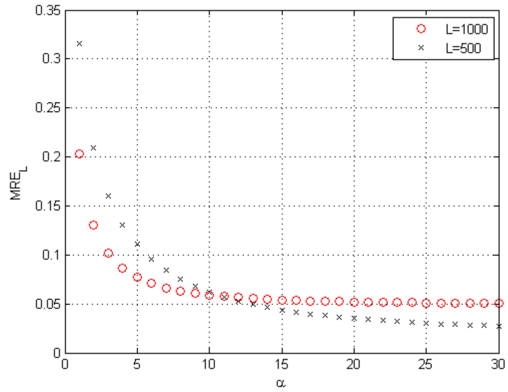

Fig. 8 Dependence of mean magnitude error on parameter α for L=500 and L=1000

5. Time Domain Behaviour

To check the influence of the decreased error at low frequencies, the equiripple relative error design needs to be compared with the optimized MRE_L. For L=500 differentiator's features are given in Fig. 9.

Compared to results from Fig. 4 one could notice a lower relative error and phase error of the differentiator at low frequencies, in particular for $\omega < 0.3\pi$. To check the time domain behaviour, both differentiators will get a triangle input signal with the unity slope. Average deviations in the time domain:

$$E_T(M) = \frac{1}{M}\sum_{n=0}^{M-1}\left|y_d[n] - y_{Id}[n-\tau]\right| , \tag{29}$$

from the ideal output y_{Id} are calculated. The parameter τ is introduced to take into consideration the delay of the output signal compared to an ideal case. The input signal and corresponding output signals for M = 40 are given in Fig. 10. The differentiator with the equiripple relative error has $E_T(40) = 0.1033$ but minimized MRE_{500} leads to $E_T(40) = 0.0584$.

The short period signal has high frequency components. Both designed filters have a significant distortion of a magnitude characteristic at high frequencies. It is the reason for similar E_T values for M=2. The longer period shift spectrum components toward low frequencies where the differentiator designed by the MRE_{500} optimization has advantages (the approximation error is lower in that range). The longer the period M the richer the spectrum at low frequencies.

In order to highlight the performances of the proposed solution, experiments are carried out with other known methods [2-6]. The results are shown in the Figs. 11-13, and Tables 3-9.

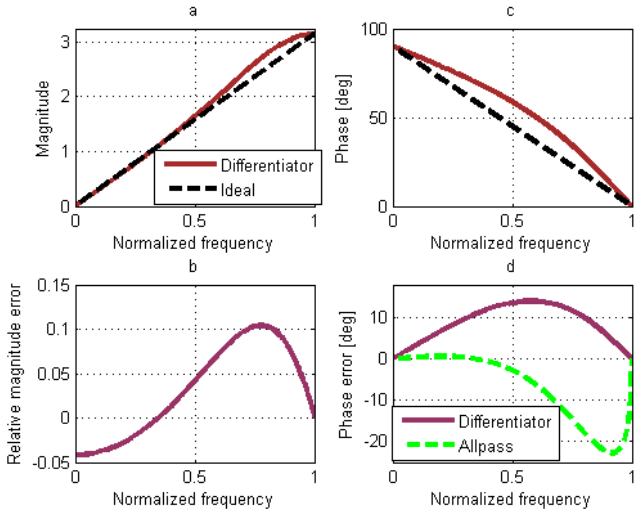

Fig. 9 (a) Magnitude, (b) relative error, (c) phase and (d) phase error of first order differentiator with minimized MRE_{500}

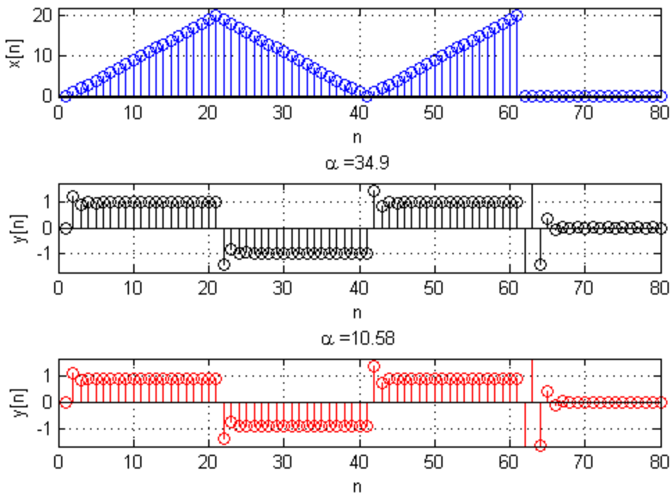

Fig. 10 Output of the differentiator for $\alpha = 34.9$ (optimized MRE_{500}) and $\alpha = 10.58$ (equiripple relative error)

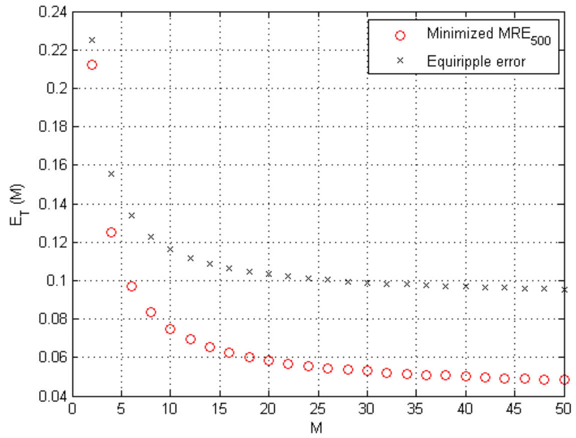

Fig. 11 Average deviation of differentiator's output from ideal output in time domain

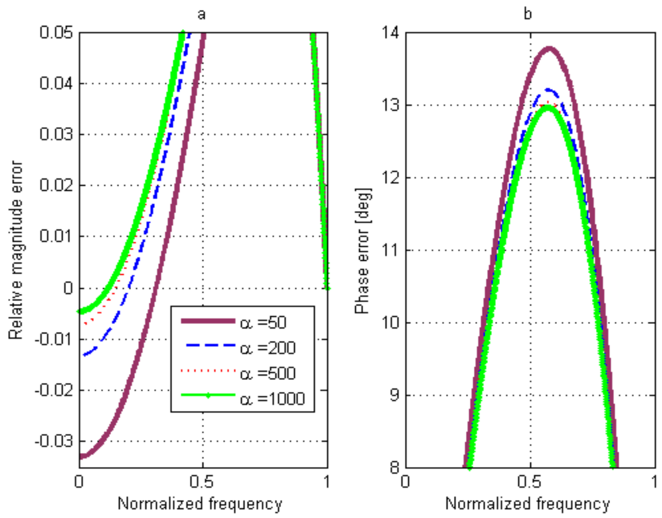

Fig. 12 Influence of parameter α on relative magnitude error and phase error of differentiator

The restricted number of bits for coefficient representations reduces the number of possible locations of filter poles. All analysed filters are robust and remain stable with only two bits dedicated for fractional parts. The given Tables show that the proposed filter has the minimum maximal relative magnitude error compared to other analysed filters if fractional parts of coefficients are represented with 2 to 8 bits. Al Aloui differentiator [3, 4] has the output signal closer to an ideal one thanks to a similar magnitude error but twice smaller phase error. A similar conclusion can be made with Goswami *et. all.* differentiator [2]. Tsai and Chu [6], and Sankranti *et. all.*, differentiators [18] have an almost double magnitude error with somewhat better phase response but increased time domain error. All given examples point to the fact that the good time domain behaviour demands not only the minimum magnitude error but also the restricted phase error.

Table 3 Parameters of differentiator designed by minimization of MRE_{500}, α=34.9, and E_T = 0.0584 after quantization

bit	δ_r	$E_{Tq}(40)$	δ_p [deg]	a_1
8	0.1041	0.0583	14.0157	0.2422
7	0.1041	0.0583	14.0157	0.2422
6	0.0994	0.0730	14.4775	0.2500
5	0.0994	0.0730	14.4775	0.2500
4	0.0994	0.0730	14.4775	0.2500
3	0.0994	0.0730	14.4775	0.2500
2	0.0994	0.0730	14.4775	0.2500

Table 4 Parameters of Goswami et. all. [2] differentiator after quantization

bit	δ_r	$E_{Tq}(40)$	δ_p [deg]	pole
8	0.1261	0.0433	9.9583	0.1729
7	0.1261	0.0433	9.9583	0.1729
6	0.1360	0.0368	9.4495	0.1642
5	0.1040	0.0384	10.4757	0.1818
4	0.1360	0.0595	10.1642	0.1765
3	0.1815	0.0107	7.1808	-0.1250
2	0.1665	0.0250	14.4775	-0.2500

Table 5 Parameters of Al Aloui differentiator [3, 4] after quantization

bit	δ_r	$E_{Tq}(40)$	δ_p [deg]	pole
8	0.1512	0.0125	8.2132	0.1429
7	0.1512	0.0125	8.2132	0.1429
6	0.1512	0.0125	8.2132	0.1429
5	0.1512	0.0125	8.2132	0.1429
4	0.1512	0.0125	8.2132	0.1429
3	0.1512	0.0125	8.2132	-0.1429
2	0.2000	0.1980	14.4775	-0.2500

Table 6 Parameters of Tsai and Chu [6] differentiator after quantization

bit	δ_r	$E_{Tq}(40)$	δ_p [deg]	pole
8	0.1676	0.1360	9.4428	0.1641
7	0.1711	0.1394	9.4428	0.1641
6	0.1730	0.1323	9.8969	0.1719
5	0.1699	0.1466	8.9893	0.1563
4	0.1790	0.1184	10.8069	0.1875
3	0.1186	0.1204	7.1808	-0.1250
2	0.1665	0.0250	14.4775	-0.2500

Table 7 Parameters of Sankranti et. all. [18] differentiator after quantization

bit	δ_r	$E_{Tq}(40)$	δ_p [deg]	pole
8	0.2005	0.1899	8.1855	-0.1424
7	0.1983	0.1865	8.2685	-0.1438
6	0.2027	0.1933	8.1022	-0.1409
5	0.2001	0.1857	8.4338	-0.1467
4	0.2058	0.2011	7.7664	-0.1351
3	0.1730	0.1480	9.0847	-0.1579
2	0.3017	0.3097	6.3794	-0.1111

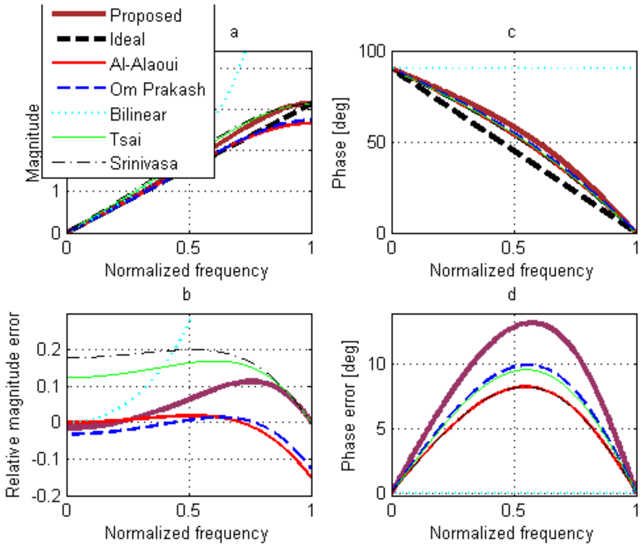

Fig. 13 Comparison of proposed with existing first order differentiators. (a) Magnitude and (b) relative magnitude error, (c) phase and (d) phase error.

Table 8 Parameters of obtained results

Diff	δ_r	*MRE*	MRE_{500}	δ_p [deg]	*MPE*	MPE_{500}	pole
Prop 34.9	0.1041	0.05052	0.02539	14.0178	8.88524	7.82163	-0.24222
Prop 500	0.1152	0.0550	0.0221	13.0299	8.2628	7.3418	-0.2255
Al Aloui [3, 4]	0.1503	0.0242	0.0082	8.2132	5.3126	5.0253	-0.1429
Goswami *et. all.* [2]	0.1265	0.0238	0.0183	9.9164	6.3924	5.9354	-0.1722
Tsai and Chu [6]	0.1693	0.1342	0.1390	9.5553	6.1636	5.7452	-0.166
Sankranti *et. all.* [18]	0.2001	0.1661	0.1877	8.2132	5.3126	5.0253	-0.1428

Table 9 Parameters of differentiators

Differentiator	Numerator	Denominator
Al Aloui [3, 4]	[1 -1]	[7/8 1/8]
Goswami *et. all.* [2]	[1.18 -1.18]	[1.04 0.1791]
Tsai and Chu [6]	[1.31 -1.31]	[1 0.166]
Sankranti *et. all.* [18]	[π -π]	[7/3 1/3]

In general, the transfer function of the first order IIR filter requires three multipliers. However, it is well known that all-pass filters, being a special case of IIR filters since numerator and denominator polynomials are mirror image polynomials, can be realized with a reduced number of multiplications which is equal to the filter order. Hence, proposed first order differentiators require only two multipliers

6. Conclusions

The design of the first order differentiator obtained by the parallel connection of direct path and IIR all-pass filter is proposed in this paper. First order differentiator could be realized with a minimum number of multiplications thanks to the applied IIR all-pass sub-filter which demands only one multiplier. The average group delay of the proposed filters equals 0.5. To provide an ideal magnitude, the all-pass phase response should deviate from the ideal one for nonlinear term of value $2\arcsin(\omega/\pi)$. This phase constraint causes the differentiators' maximum phase error to be above 13 degrees but this maximum occurs at high frequencies where in a real situation scenario the input signal spectral components are not significant. This feature could provide a good time domain behaviour. The proposed approach could provide the equiripple magnitude and phase error but this is not the optimal solution if the time domain behaviour is of primary importance. To minimize the deviation of the output signal it is important to provide as minimum as possible both phase and magnitude errors at low frequencies. It is achieved by introducing the parameter defining the ratio of the second and first all-pass filter's phase error considering that a straightforward dependence between all-pass filter phase and differentiator magnitude exists. Minimum number of multiplications required for filter realization makes them ideal candidates for VLSI and real-time applications. The finite word-length analysis shows that the proposed differentiator remains stable even when the coefficient's fractional part is represented by only two bits.

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